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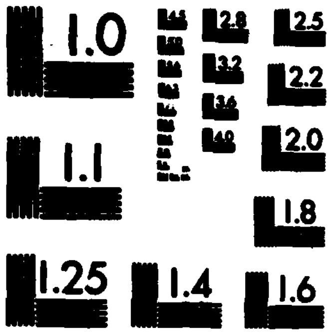

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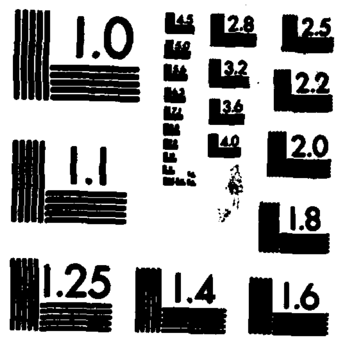

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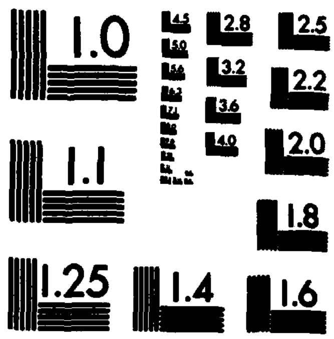

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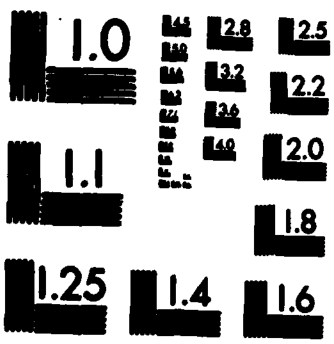

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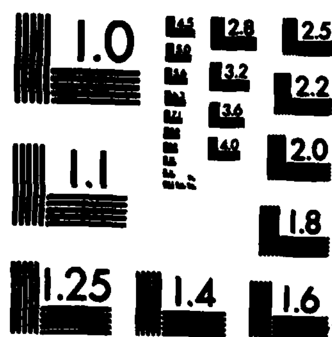

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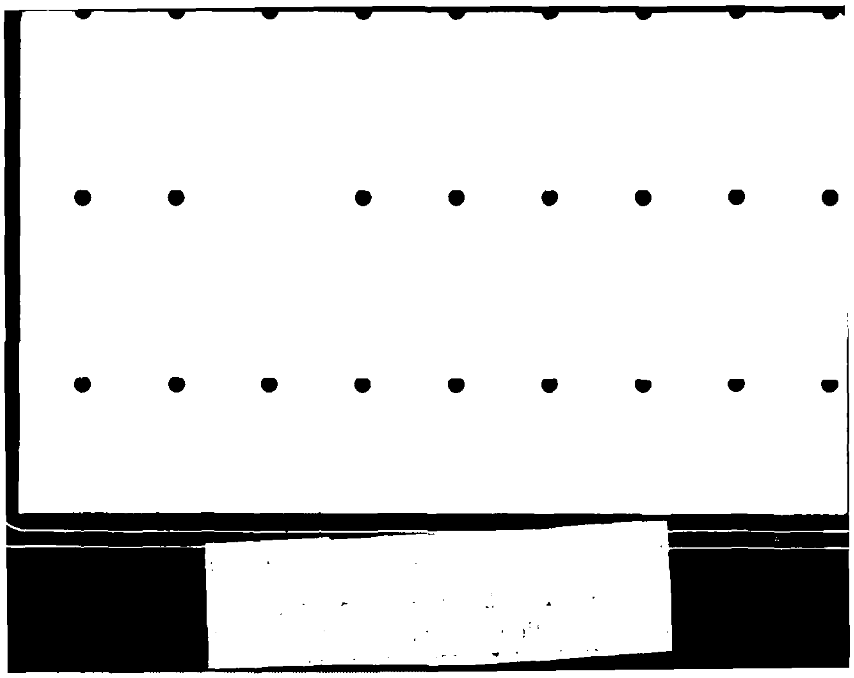

LIMIT CYCLES OF PLANAR

QUADRATIC DIFFERENTIAL EQUATIONS

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D. E. Koditschek and K. S. Narendra

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Limit Cycles of Planar Quadratic Differential Equations

D. E. Koditschek and K. S. Narendra

Introduction:

Since Hilbert posed the problem of systematically counting and locating the limit cycles of polynomial systems on the plane in 1900, much effort has been expended in its investigation. A large body of literature -- chiefly by Chinese and Soviet authors -- has addressed this question in the context of differential equations whose field is specified by quadratic polynomials. In this paper we consider the class of quadratic differential equations which admit a unique equilibrium state, and establish sufficient conditions for the existence and uniqueness of limit cycles. The work is based upon insights and techniques developed in an earlier analysis of such systems [1] motivated by questions from mathematical control theory.

Until the fifties, work on quadratic systems chiefly concerned the existence of a center. In 1952, Bautin [2] showed that a given equilibrium state can support as many as but no more than three limit cycles under a quadratic field. Three years later, a paper by Petrovskii and Landis [3] purported to show that a quadratic system could support no more than three cycles on the entire plane. Although this result was called into question by several researchers (and the authors later acknowledged an error in the proof [4]) it apparently inspired a number of attempts over the next decade to complete the Hilbert program for quadratic differential equations [5,6,7]. Since Coppel's useful survey [8], to our knowledge there have been only two contributions to this problem. Perko [9] has demonstrated the existence of limit cycles for a certain class of quadratic systems, and, recently, Shi Songling [10] has presented a quadratic system which apparently has four limit cycles, settling the question of the validity of [3] negatively. Thus, the Sixteenth Hilbert Problem remains unsolved even for quadratic systems.

By a "quadratic system" is meant the differential equation

$$\dot{x} = Ax + \begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix} \tag{1}$$

where $A,G,H \in \mathbb{R}^{2x2}$ (and $x^T Gx$ denotes the scalar product of the vectors x and $Gx \in \mathbb{R}^2$). We adopt the convention

$$B(x) \triangleq \begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix} .$$

Our results may be summarized briefly as follows. In section 2 it is shown (Lemma 1) that any quadratic system with a unique equilibrium state may be written in the form

$$\dot{x} = Ax + c^T xDx \tag{2}$$

where $c \in \mathbb{R}^2$, $D \in \mathbb{R}^{2x2}$, and (Lemma 2) that the global boundedness of all solutions depends upon the spectrum of the pencil $A + \mu D$, when μ takes values in the range of a specified nonlinear functional. In section 3, Theorem 2 establishes the existence of limit cycles based upon the spectrum of A and $A + \mu D$ as an easy extension of the previous results. Theorem 3, in section 3, uses a functional specifying the eigenvalues of $A + \mu D$ to demonstrate that such systems have only one limit cycle.

2. Preliminary Discussion of Techniques:

As mentioned earlier, the techniques of [1] are used extensively in this paper, and are introduced here for ease of exposition. We will use the following notation throughout the paper: if $x, y \in \mathbb{R}^2$, then $x^T y$ denotes the scalar product of x and y; $|x,y|$ denotes the determinant of the array formed by the coordinates of x and y. The skew symmetric matrix $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a 90° rotation and the orthogonal complement of x is denoted by $x_\perp \triangleq Jx$. We note that

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$|x,y| = y^T x_\perp = y^T Jx.$

A line through the origin containing x is denoted $\langle x\rangle \overset{\Delta}{=} \{y : y = \alpha x,\ \alpha \in \mathbb{R}\}$. If $A \in \mathbb{R}^{2x2}$, the symmetric part of A is denoted $A_s = 1/2[A + A^T]$. If A is singular it can be written as the outer product $A = ab^T$ where a, $b \in \mathbb{R}^2$. The zeros of the quadratic form $x^T Ax = x^T A_s x$ then lie on $\langle a_\perp\rangle \cup \langle b_\perp\rangle$. Conversely, if for some $a \in \mathbb{R}^2$, $x^T Ax$ is zero on $\langle a_\perp\rangle$ then $A = [ab^T]_s$, for some $b \in \mathbb{R}^2$.

By limiting attention to quadratic systems with a single equilibrium state, the structure of (1) is greatly simplified. This is a consequence of the fact that A and B are homogeneous polynomials in x, hence map subspaces of $\mathbb{R}^2$ into subspaces.

Lemma 1: If A is bijective then system (1) has a unique equilibrium state only if there exists a $c \in \mathbb{R}^2$ and a $D \in \mathbb{R}^{2x2}$ such that $B(x) = c^T xDx$.

Proof: If for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$, $\lambda Ax_0 = B(x_0) = 0$ then $-1/\lambda\ x_0$ is an equilibrium state of system (1). Hence, we require that $|Ax, B(x)| = 0$ imply $B(x) = 0$ for any $x \neq 0$. Since $|Ax,B(x)| = x_1^3 q(x_2/x_1)$, where q is a cubic polynomial in x_2/x_1, there exists at least one real zero of q, (say ν_0). Hence the system has a unique equilibrium state only if B = 0 on $\langle \begin{bmatrix} 1 \\ \nu_0 \end{bmatrix} \rangle$. This, in turn, implies that both quadratic forms in B share a common zero line, or $G = [cd_1^T]_s$, $H = [cd_2^T]_s$ where $c = \begin{bmatrix} 1 \\ \nu_0 \end{bmatrix}_\perp$ and $d_1, d_2 \in \mathbb{R}^2$. Defining $D = \begin{bmatrix} d_1^T \\ d_2^T \end{bmatrix}$, we have the desired result.

□

Since it has been shown [8] that no limit cycles of (1) may occur unless the linear part of the field has complex conjugate eigenvalues, Lemma 1 achieves an immediate simplification of the problem. Any quadratic system with a unique equilibrium state capable of supporting a limit cycle must admit the expression given in (2).

Representing a quadratic system in the form (2) affords analysis by well known methods of linear algebra. The following lemma relates the properties of the pencil $A + \mu D$ to the interaction of the quadratic and linear part of the field in (2), providing the basis for much of our global analysis.

<u>Lemma 2</u>: For any matrices $A, D \in \mathbb{R}^{2\times 2}$ and any $x \in \mathbb{R}^2$, if $\mu(x) \triangleq -\frac{|Ax,x|}{|Dx,x|}$ is defined, then x is an eigenvector of the pencil $A + \mu(x)D$ with corresponding eigenvalue $\lambda(x) \triangleq -\frac{|Ax,Dx|}{|Dx,x|}$.

<u>Proof</u>: Define $\alpha(x) \triangleq |Ax,x|$ and $\delta(x) \triangleq |Dx,x|$. Since $|[\delta A-\alpha D]x,x| = \delta|Ax,x| - \alpha|Dx,x| = 0$ for all $x \in \mathbb{R}^2$, it follows that $[\delta A-\alpha D]x = \eta(x)x$ for some real valued function η. But $x^T x\eta = x^T[\delta A-\alpha D]x = \begin{vmatrix} x^T Ax & x^T Dx \\ x^T JAx & x^T JDx \end{vmatrix} = |[x,J^T x][Ax,Dx]| = -x^T x|Ax,Dx|$. Hence $\eta(x) = -|Ax,Dx|$ and the result follows. □

<u>Corollary 2.1</u>: Let A have complex-conjugate eigenvalues. If the pencil $A + \mu D$ is bounded and non-singular then system (2) has a unique equilibrium state at the origin.

<u>Proof</u>: If A has no real eigenvectors then $\mu(x)$ cannot be bounded unless $|Dx,x|$ is sign definite - i.e. D has no real eigenvectors. According to Lemma 2 $A + \mu D$ is nonsingular if $\lambda(x) \neq 0$; since $|Dx,x|$ is sign definite, this is equivalent to the condition $|Ax,Dx|$ is sign definite. The latter assures that $|Ax,B(x)| = 0$ only on $\langle c_\perp \rangle$ where $B \equiv 0$. Thus the origin is the unique equilibrium state. □

<u>Corollary 2.2</u>: Let A have complex-conjugate eigenvalues. If the pencil $A + \mu D$ is bounded and has negative real eigenvalues over the range of $\mu(x)$ then system (2) has no unbounded solutions.

<u>Proof</u>: As in the previous corollary the fact that μ is bounded immediately implies that D has complex-conjugate eigenvalues. If, additionally, the pencil has a spectrum on the left half of the real line in C, then the sign of $|Ax,Dx|$ must agree with the sign definite form $|Dx,x|$. We will now interpret this sign agreement geometrically to show that the spiral curve defined by a single loop of the linear trajectory $e^{tD}y$ defines a positive-invariant region in the phase plane, arbitrarily far from the origin.

Choose a point, y, on $\langle c\rangle$ whose sign is opposite to the sign of the real part of the eigenvalues of D, say on the positive ray. Let $\Delta \triangleq \{e^{tD}y \mid t \in [0,t^*];\ e^{t^*D}y = \gamma y;\ 0 < \gamma < 1\}$ be a complete spiral loop and let $\Lambda \triangleq \{\zeta y \mid \zeta \in [\gamma,1]\}$ join its end-points as depicted in Figure 1.

The normal to the curve at any point $x \in \Delta$ lies in $\langle JDx\rangle$ and since $x^T JDx = |Dx,x|$, JDx is either interior directed or exterior directed, depending upon whether $|Dx,x|$ is negative or positive, respectively. With no loss of generality, we assume $|Dx,x| < 0$, hence JDx is the interior directed normal to Δ at x. Similarly, Jy is the interior directed normal to Λ for any $y \in \Lambda$. It now suffices to show that $f^T(x)JDx > 0$ for $x \in \Delta$, and $f^T(y)Jy > 0$ for $y \in \Lambda$.

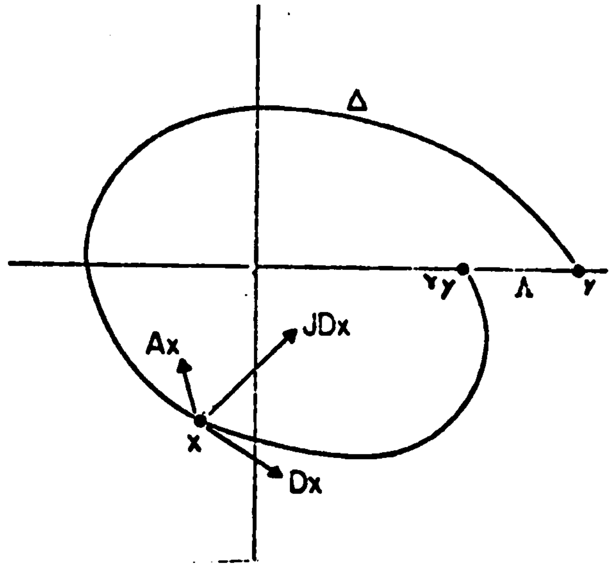

Figure 1

Expanding the first inequality, we have

$$f^T JDx = x^T[A^T + c^T x D^T]JDx = x^T A^T JDx$$

$$= |Dx,Ax| = -|Ax,Dx| > 0$$

for all $x \in \mathbb{R}^2$. Expanding the second inequality, we have

$$f^T Jy = -y^T Jf = -y^T JAy - c^T y\, y^T JDy$$

$$= -|Ay,y| - c^T y |Dy,y|$$

hence, because $c^T y > 0$ for $y \in \Lambda$, and $|Dy,y| < 0$, the desired inequality holds when the second term dominates the first term far enough away from the origin.

$\square$

<u>Corollary 2.3</u>: Let A have complex conjugate eigenvalues. If the pencil $A + \mu D$ is bounded and has positive real eigenvalues over the range of $\mu(x)$, then the system (2) has unbounded solutions for every initial condition outside a compact neighborhood of the origin.

<u>Proof</u>: Let y be a point on <c> whose sign is the same as the sign of the real part of the eigenvalues of D, say on the negative ray. Let Δ and Λ be as in the proof of corollary 2.2, depicted in Figure 1. Assume again with no loss of generality that JDx is the interior directed normal to Δ at x and Jy the interior normal to Λ for $y \in \Lambda$. We need to show that $f^TDx < 0$ for $x \in \Delta$ and $f^TJy < 0$ for $y \in \Lambda$. Since $|Ax,Dx|$ has opposite sign to $|Dx,x|$ under the assumption that the pencil has positive real eigenvalues, the first inequality follows for every spiral loop Δ. The second inequality holds on Λ outside of the last loop for which $|\gamma c^T y|$ is less than the constant $\left| \frac{y^TJAy}{y^TJDy} \right|$. □

The reader should observe that the conditions of these corollaries may be tested using standard computations involving the elements of A and D as established in the proofs. To summarize, the condition that a matrix, $M \in \mathbb{R}^{2x2}$, have complex conjugate eigenvalues is equivalent to the condition that $|Mx,x|$ never vanish on $\mathbb{R}^2$, or, equivalently, that $[JM]_s$ be a positive or negative definite matrix. If A has complex conjugate eigenvalues, then $\mu(x)$ is bounded if and only if $|Dx,x|$ has no zeros, or, equivalently, D has complex conjugate eigenvalues as well. Under these conditions the sign definiteness of $\lambda(x)$ depends entirely upon the sign definiteness of $|Ax,Dx| = |D|\cdot|D^{-1}Ax,x|$, hence, upon whether $D^{-1}A$ has complex conjugate eigenvalues, or, using the identity in $\mathbb{R}^{2x2}$, $D^{-1} = J^TD^TJ$, whether $[D^TJA]_s$ is a positive or negative definite matrix.

It is worth noting, in passing, that the conditions for the boundedness of system (2) resulting from Lemma 2 and its corollaries may be extended to cover

any planar quadratic system (1). Earlier work [1],[11],[12] has established that homogeneous quadratic systems which may not be written in the form $c^T xDx$ must be unstable. It may be shown [1],[13], in consequence, that system (1) must have unbounded solutions if it cannot be written in the form (2). Imposing the added condition that the linear part of the field not be unstable, and adjusting for special cases permits the following characterization of any globally asymptotically stable quadratic differential equation.

Theorem 1 (Koditschek and Narendra [1]): System (1) is globally asymptotically stable if and only if

(i) The eigenvalues of A have non-positive real part;

(ii) There exist a $c \in \mathbb{R}^2$ and $D \in \mathbb{R}^{2x2}$ such that $B(x) = c^T xDx$;

(iii) The pencil $A + \mu(x)D$, where $\mu(x) \triangleq - \frac{|Ax,x|}{|Dx,x|}$, has non-positive eigenvalues for all $x \in \mathbb{R}^2$; is unbounded, if ever, only on an eigenvector of A in the null space of D with $|A| \neq 0$; is singular on at most a unique line which coincides with $<c_\perp>$ iff $|A| \neq 0$.

In fact, according to [13], conditions (ii) and (iii) of this theorem (if we relax the stipulation that $|A| \neq 0$ when the pencil is unbounded) are necessary and sufficient for the boundedness of solutions to any quadratic system (1) as well.

However, in the sequel, we will confine our attention to quadratic systems of the form (2), and, specifically, those guaranteed by Corollary 2.1 to have a single equilibrium state.

3. Existence and Uniqueness of Limit Cycles:

We now apply the techniques developed in the previous section to the subject proper - an account of the limit cycles of system (2). While those results will be seen to establish the existence conditions directly, the proof of uniqueness involves further development.

According to the results of Lyapunov, the local stability behavior of system (2) is entirely determined by the linear part of the field when the matrix A has

eigenvalues with non-zero real part. On the other hand, the global arguments of Corollaries 2.2 and 2.3 depend upon the spectrum of the pencil $A + \mu D$. Using the latter result to guarantee boundedness, and assuming local instability by the former argument, we take advantage of the special nature of limit sets of planar dynamical systems established by the Poincaré-Bendixson Theorem to conclude that a limit cycle must exist.

<u>Theorem 2</u>: System (2) has a limit cycle if

(i) the matrix A has complex conjugate eigenvalues with non-zero real parts

and (ii) the pencil $A + \mu D$ is bounded with non-zero eigenvalues opposite in sign to the real part of the eigenvalues of A over the range of $\mu(x)$.

<u>Proof</u>: Assume that (i) holds, and the eigenvalues of A have positive real parts. Then the origin is totally unstable, hence for some positive definite symmetric matrix, P, $\mathbb{R}^2 - \{x | x^T P x < \gamma\}$ for any $\gamma > 0$ is a positive invariant set of system (2). If $A + \mu(x)D$ is bounded with non-zero eigenvalues, then the origin is the sole critical point of system (2), according to Corollary 2.1. By Corollary 2.2, if the eigenvalues of $A + \mu(x)D$ are negative, then all solutions of (2) are bounded: in particular, the Jordan Curve $\Delta \cup \Lambda$ bounds a positive-invariant set, J, containing the origin. Thus $J - \{x^T P x < \gamma\}$ is a compact positive-invariant set, free of critical points. In consequence of the Poincaré-Bendixson Theorem, the positive limit set of a trajectory in $J - \{x^T P x < \gamma\}$ must be a limit cycle [14, p. 232, Thm. 9.3].

If the eigenvalues of A have negative real part and the eigenvalues of $A + \mu(x)D$ are positive, an identical argument concerning negative limit sets using Corollary 2.3 will establish the existence of a limit cycle. □

While the question of necessity is not formally addressed in this paper, it is useful to remark upon the existence of limit cycles when the conditions of Theorem 2 are not met. Assuming (ii), Condition (i) is certainly necessary for systems of the form (2) to support a limit cycle: Coppel [8] has shown that A must have complex conjugate eigenvalues; when A has purely imaginary eigenvalues

and $A + \mu(x)D$ has negative eigenvalues Theorem 1 guarantees global asymptotic stability, while a similar argument establishes that all non-zero solutions of (2) grow without bound when $A + \mu(x)D$ has positive eigenvalues in this case. If (i) holds and $A + \mu(x)D$ has a zero eigenvalue for some $x \in \mathbb{R}^2$ then system (2) has at least one critical point distinct from the origin. If (i) holds and $A + \mu(x)D$ is bounded with non-zero eigenvalues whose signs agree with the real part of $\sigma(A)$ then system (2) is either globally asymptotically stable according to Theorem 1, or can be shown to admit only unbounded non-zero solutions. However, if (i) holds, the spectrum of $A + \mu(x)D$ is sign definite, but $\mu(x)$ is not bounded, then D has real eigenvalues and the possibility of a limit cycle remains. As will be seen below, there is good reason to suspect that system (2) cannot support a limit cycle unless D has complex conjugate eigenvalues. If true, this would imply that the conditions of Theorem 2 are both necessary and sufficient for a quadratic system (1) with a single critical point to support a limit cycle.

We now proceed to show that the limit cycle established by Theorem 2 is indeed unique. Along the way we will restate the conditions of that theorem (Lemma 3, below) and provide a better intuitive sense of the mechanism underlying the isolated periodic solution. This is achieved by a transformation to polar coordinates.

Assuming A has complex conjugate eigenvalues we may always find a coordinate system (under linear transformation of the state) such that $A = \sigma I + \omega J$ – where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\sigma, \omega \in \mathbb{R}$ – and $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, defining the polar coordinate transformation $\rho \triangleq [x_1^2 + x_2^2]^{1/2}$, $\theta \triangleq \arctan x_2/x_1$, equation (2) may be written as

$$\begin{aligned} \dot{\rho} &= 1/\rho \, f^T(x)x = \rho[\sigma + \rho d(\theta)] \\ \dot{\theta} &= \frac{f^T(x)Jx}{x^T x} = \omega + \rho\bar{d}(\theta) \end{aligned} \tag{3}$$

where d and $\bar{d}$ are functions of θ only and are defined by

$$d(\theta) \triangleq \cos\theta \frac{x^T D x}{x^T x} \; ; \; \bar{d}(\theta) \triangleq \cos\theta \frac{x^T D^T J x}{x^T x} \quad .$$

Under the assumptions of Theorem 2, $\bar{d}$ is sign definite for $\theta \in [-\pi/2, \pi/2]$ and we assume, with no loss of generality, that sgn ω = sgn $\bar{d} > 0$. We define $\eta(\theta) \triangleq \sigma\bar{d}(\theta) - \omega d(\theta)$ and assert the following.

Lemma 3: The following conditions are equivalent to those stated in Theorem 2, and hence are sufficient for the existence of a limit cycle of system (3):

$$\begin{array}{ll} \text{either (i)} & \sigma > 0 \text{ and } \eta < 0 \\ \text{or (ii)} & \sigma < 0 \text{ and } \eta > 0 \end{array} \quad \text{for } \theta \in [-\pi/2, \pi/2]$$

Proof: Since $A = \sigma I + \omega J$, condition (i) of Theorem 2 is equivalent to one of the sign conditions on σ. From Lemma 2, the eigenvalues of $A + \mu(x)D$ are given by

$$\lambda(x) = \frac{-|Ax, Dx|}{|Dx, x|} = -\frac{1}{|Dx, x|} (\sigma|x, Dx| + \omega|Jx, Dx|) = \frac{\eta}{\bar{d}} \; .$$ Thus, for $\theta \in [-\pi/2, \pi/2]$, the sign conditionson $\bar{d}$ and η are equivalent to condition (ii) of Theorem 2. □

As reported in [8], limit cycles of quadratic differential equations enclose convex regions, hence, any periodic solution of (3) must have an angular derivative, $\dot{\theta}$, of constant sign: no limit cycle may leave the region $C \triangleq \{x \in \mathbb{R}^2 | \omega + \rho\bar{d} > 0\}$. Consider $x(t;p_0)$, a trajectory in C originating at p_0 - a point on the negative x_2-axis. For some $t_2 > t_1 > 0$ we must have $x(t_1;p_0) = p_1$ - a point on the positive x_2-axis; and $x(t_2;p_0) = p_2$ - a point on the negative x_2-axis - as depicted in Figure 2. Denote the resulting curve in the right half plane over the interval $[0,t_1]$ as Γ_1, and the left half plane curve, over the interval $[t_1,t_2]$, as Γ_2. Evidently, Γ_2 may be expressed as $x(s;p_1)$ where $s \in [0,t_2-t_1]$. Now map Γ_1 into Γ_2 as follows: since $\dot{\theta}$ is sign-definite, for every $t \in [0,t_1]$ there exists a unique $s \in [0,t_2-t_1]$ and $\zeta > 0$ such that

$$x(s;p_1) = -\zeta x(t;p_0) .$$

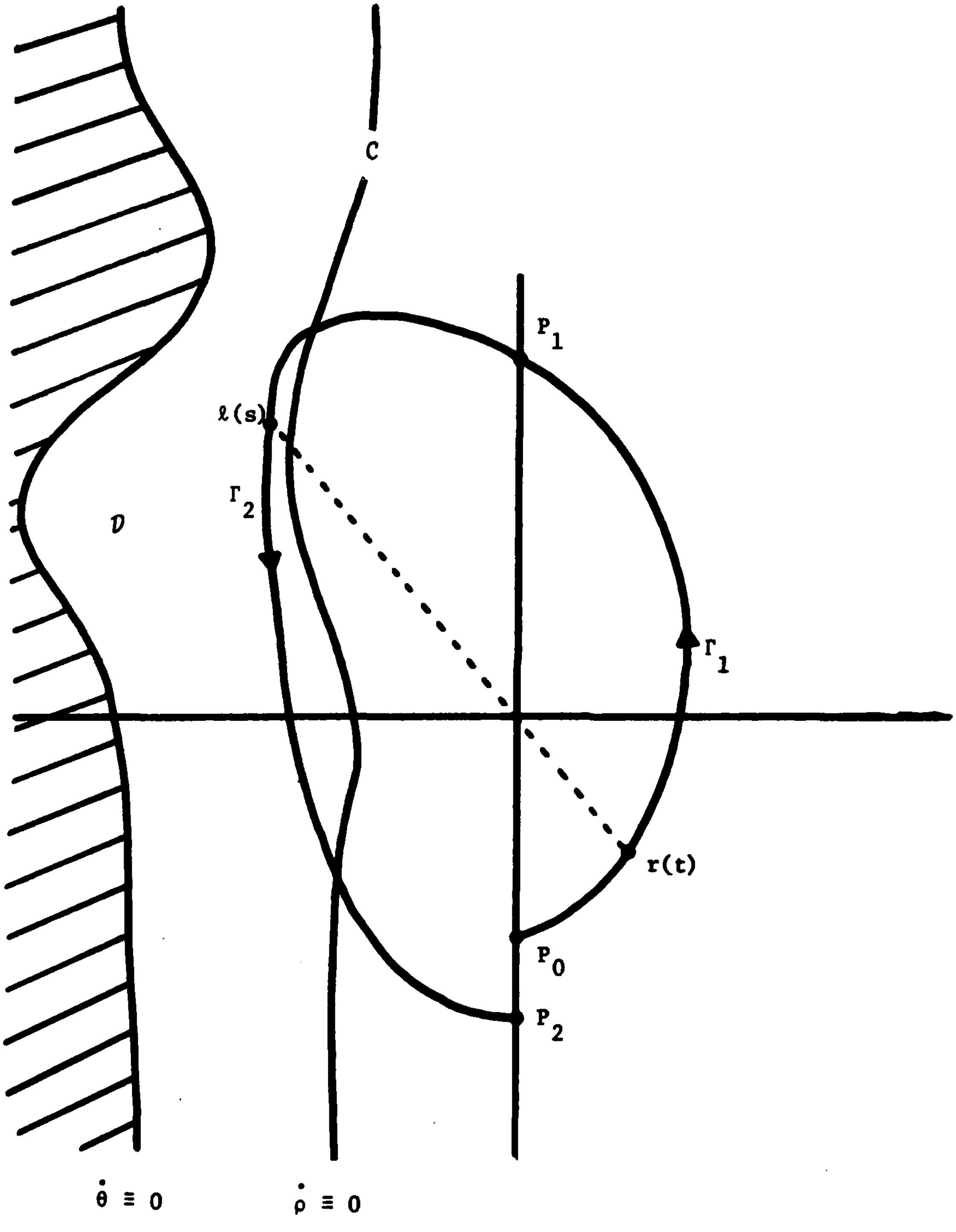
C
P_1
$\ell(s)$
Γ_2
$\mathcal{D}$
Γ_1
$r(t)$
P_0
P_2
$\dot{\theta} \equiv 0$
$\dot{\rho} \equiv 0$

Figure 2

For convenience we shall denote points on the right hand curve, Γ_1, by $r(t)$, and on the left hand curve, Γ_2, by $\ell(s)$, letting $\rho \overset{\Delta}{=} \|r(t)\|$ and $\lambda \overset{\Delta}{=} \|\ell(s)\| = \zeta\rho$. The chief advantage of this map is the induced functional dependence of s on t, hence the ability to write a differential equation for ρ and λ using the same angular interval. From (4) and the above, we have, for fixed initial conditions,

$$\begin{aligned} \frac{d}{d\theta} \ln \rho &= \frac{\sigma + \rho d}{\omega + \rho \bar{d}} \\ & & \theta \in [-\pi/2, \pi/2]. \qquad (4) \\ \frac{d}{d\theta} \ln \lambda &= \frac{\sigma - \lambda d}{\omega - \lambda \bar{d}} \end{aligned}$$

The restatement of Theorem 2 in Lemma 3 lends added insight into the mechanism by which $x(t;p_0)$ grows and decays on $\Gamma_1 \cup \Gamma_2$. Considering condition (i) of Lemma 3, since $\sigma\bar{d} > 0$ on $[-\pi/2, \pi/2]$, the fact that $\eta < 0$ necessitates $d > 0$ on that interval. Hence, from (4), while ρ must increase on Γ_1, $\dot{\lambda}$ becomes negative when Γ_2 enters the region $\mathcal{D} \overset{\Delta}{=} \{x \in R^2 | x_1 < -\sigma \frac{x^T Dx}{x^T x}\}$ in the left half plane. Moreover, C has a boundary, ∂C, in the left half plane and $\eta < 0$ implies $\partial C \subseteq \mathcal{D}$ - i.e. that certain trajectories contained in C must enter $\mathcal{D}$. Since $\frac{d}{d\theta} \lambda \to -\infty$ as $\ell(s) \to \partial C$, the growth of a trajectory on Γ_1 is countered with increasing effect on a portion of Γ_2, resulting in a limit cycle. Notice that if D has real eigenvalues then $\bar{d}$ is no longer sign definite, hence d may not be sign definite, and these remarks are no longer valid, underscoring the importance of the requirement that D have complex conjugate eigenvalues.

The differential equations in (4) define two families of functions, $\rho(\theta;\rho_0)$ and $\lambda(\theta;\lambda_0)$, parametrized by initial condition on the negative and positive x_2-axes, respectively. Observing that $\lambda_0 = \rho(\pi/2;\rho_0)$ and that the vector fields in (4) are smooth when $x \in C$, we may explicitly regard ρ and λ as functions of θ and ρ_0, continuously differentiable in both arguments. Since distinct integral curves of

autonomous systems defined by smooth fields remain distinct over all times, we have $\frac{\partial}{\partial\rho_0}\rho > 0$ and $\frac{\partial}{\partial\rho_0}\lambda > 0$ for all $\theta \in [-\pi/2,\pi/2]$. Hence, the function

$$\psi(\rho_0) \triangleq \frac{\lambda(\pi/2,\rho_0)}{\rho(-\pi/2,\rho_0)} ,$$

which represents the ratio of the magnitudes of the end points of the curve $\Gamma_1 \cup \Gamma_2$ (both on the negative x_2-axis) is a continuously differentiable function of ρ_0. Evidently, $\Gamma_1 \cup \Gamma_2$ is the integral curve of a limit cycle if and only if $\psi = 1$. The proof of uniqueness involves a demonstration that ψ is monotone in ρ_0 over an interval of interest, and hence, may pass through 1 at most once. That demonstration depends upon the following computation.

<u>Corollary 3.1</u>: Conditions (i) and (ii) of Lemma 3 respectively imply

$$\text{(i)} \quad \frac{\partial}{\partial\theta} \ln \left(\frac{\partial\lambda}{\partial\rho_0} \frac{\partial\rho}{\partial\rho_0} \right) < 2\left[\frac{\partial}{\partial\theta} (\ln \lambda\rho) - \sigma/\omega \right]$$

$$\text{(ii)} \quad \frac{\partial}{\partial\theta} \ln \left(\frac{\partial\lambda}{\partial\rho_0} \frac{\partial\rho}{\partial\rho_0} \right) > 2\left[\frac{\partial}{\partial\theta} (\ln \lambda\rho) - \sigma/\omega \right]$$

<u>Proof</u>: $\frac{\partial}{\partial\theta} \ln \left(\frac{\partial\lambda}{\partial\rho_0} \frac{\partial\rho}{\partial\rho_0} \right) = \frac{\partial^2\lambda}{\partial\theta\partial\rho_0} \Big/ \frac{\partial\lambda}{\partial\rho_0} + \frac{\partial^2\rho}{\partial\theta\partial\rho_0} \Big/ \frac{\partial\rho}{\partial\rho_0}$. From (4) we have

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\lambda}{\partial\theta}\right) = \left[\frac{\partial}{\partial\theta} \ln \lambda + \lambda \frac{\eta}{(\omega-\lambda\bar{d})^2} \right] \frac{\partial\lambda}{\partial\rho_0} \quad \text{and} \quad \frac{\partial}{\partial\rho_0}\left(\frac{\partial\rho}{\partial\theta}\right) = \left[\frac{\partial}{\partial\theta} \ln \rho + \rho \frac{\eta}{(\omega+\rho\bar{d})^2} \right] \frac{\partial\rho}{\partial\rho_0}$$

Since $\bar{d} > 0$, we have $\frac{\omega}{\omega - \lambda\bar{d}} > 1$, and $\frac{\omega}{\omega + \rho\bar{d}} < 1$, for all $\theta \in [-\pi/2,\pi/2]$

Hence, if condition (i) of Lemma 3 holds then

$$\lambda \frac{\eta}{(\omega-\lambda\bar{d})^2} < \frac{\lambda}{\omega} \frac{\eta}{\omega - \lambda\bar{d}} = \frac{\partial}{\partial\theta} \ln \lambda - \sigma/\omega$$

and

$$\rho \frac{\eta}{(\omega+\rho\bar{d})^2} < \frac{\rho}{\omega} \frac{\eta}{\omega + \rho\bar{d}} = \frac{\partial}{\partial\theta} \ln \rho - \sigma/\omega,$$

since $\eta < 0$ on $[-\pi/2,\pi/2]$ and substituting from (4).

Since $\frac{\partial\lambda}{\partial\rho_0}$, $\frac{\partial\rho}{\partial\rho_0} > 0$, this implies

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\lambda}{\partial\theta}\right) < \left(2\frac{\partial}{\partial\theta}\ln\lambda - \sigma/\omega\right)\frac{\partial\lambda}{\partial\rho_0}$$

$$\frac{\partial}{\partial\rho_0}\left(\frac{\partial\rho}{\partial\theta}\right) < \left(2\frac{\partial}{\partial\theta}\ln\rho - \sigma/\omega\right)\frac{\partial\rho}{\partial\rho_0}$$

yielding (i), above. The identical argument holds for (ii) with signs reversed, since $\eta > 0$. □

We may now state the second principal result of this paper.

Theorem 3: Under the conditions of Theorem 2, system (2) has only one limit cycle.

Proof: $x(t;p_0)$ is a limit cycle of (2) if and only if $\psi(\rho_0) = 1$ in system (3). According to Theorem 2, $L \triangleq \{\rho_0 > 0 | \psi(\rho_0) = 1\}$ is non-empty, and bounded away from the origin, hence $\rho_0^* \triangleq \inf L$ exists and $\rho_0^* > 0$. We will show that $\frac{d}{d\rho_0}\psi$ is sign definite for all $\rho_0 > \rho_0^*$, hence $x(t;p_0^*)$ is the only limit cycle of (2).

Note that $\frac{d}{d\rho_0}\psi = \frac{1}{\rho_0}\left[\frac{\partial}{\partial\rho_0}\lambda(\pi/2,\rho_0) - \psi\right]$. We will show below that under condition (i) of Lemma 3,

$$\frac{\partial}{\partial\rho_0}\lambda(\pi/2,\rho_0) < \psi^2 e^{\frac{-2\pi\sigma}{\omega}} ,$$

and hence $$\frac{d}{d\rho_0}\psi < \frac{\psi}{\rho_0}\left[\psi e^{\frac{-2\pi\sigma}{\omega}} - 1\right].$$

Since $\sigma/\omega > 0$ and $\psi(\rho_0^*) = 1$, this is clearly negative for $\rho_0 > \rho_0^*$. Similarly, under condition (ii) of Lemma 3 the inequalities are reversed, and $\sigma/\omega < 0$ so that $\frac{d}{d\rho_0}\psi > 0$ for all $\rho_0 > \rho_0^*$.

To obtain the bound on $\frac{\partial}{\partial\rho_0}\lambda(\pi/2,\rho_0)$ we recall that $\lambda(-\pi/2,\rho_0) = \rho(\pi/2,\rho_0)$ and $\rho(-\pi/2,\rho_0) = \rho_0$, hence

$$\ln \frac{\partial}{\partial\rho_0} \lambda\ (\pi/2,\rho_0) = \ln\left[\frac{\partial}{\partial\rho_0} \lambda\ (\pi/2,\rho_0)\ \frac{\partial}{\partial\rho_0}\ \rho\ (\pi/2,\rho_0)\right] - \ln\left[\frac{\partial}{\partial\rho_0} \lambda\ (-\pi/2,\rho_0)\frac{\partial}{\partial\rho_0}(-\pi/2,\rho_0\right.$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial\theta} \ln\ [\frac{\partial}{\partial\rho_0} \lambda\ (\theta,\rho_0)\ \frac{\partial}{\partial\rho_0}\ \rho\ (\theta,\rho_0)]d\theta$$

Applying Corollary 3.1 to case (i) yields

$$\ln \frac{\partial}{\partial\rho_0} \lambda\ (\pi/2,\rho_0) < \int_{-\pi/2}^{\pi/2} 2(\frac{\partial}{\partial\theta} \ln\ \rho\lambda\ - \sigma/\omega)d\theta = \ln\ (\ \frac{\lambda(\pi/2,\rho_0)}{\rho(-\pi/2,\rho_0)}\ )^2 - 2\pi\ \sigma/\omega$$

$$= \ln\ \psi^2 - 2\pi\ \sigma/\omega,$$

hence $\frac{\partial}{\partial\rho_0} \lambda\ (\pi/2,\rho_0) < \psi^2\ e^{\frac{-2\pi\sigma}{\omega}}$ as claimed. Case (ii) proceeds identically with the signs reversed. □

4. Conclusions:

This paper presents sufficient conditions for the existence of limit cycles of quadratic systems with a unique equilibrium state. The conditions guarantee that the limit cycle is unique. The results are based upon insights and techniques developed during an earlier investigation of the global stability properties of (1) [1], facilitated by the expression of that system in the form (2). They strongly suggest that these conditions are necessary as well, hence, that no quadratic system with a unique equilibrium state can support more than one limit cycle. That result, the uniqueness of a limit cycle around any equilibrium state of (2), the relation of limit cycles of (2) to those supported by general quadratic systems (1), all remain to be rigorously established.

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